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Spectral analysis of the stochastic Helmholtz equation in the plane

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1 Introduction

We consider time-harmonic acoustic or transverse-electric (TE) electromagnetic waves in the plane at frequency $k_0 > 0$. A source f is supported in a subset of an open, relatively compact set $\Omega \subset \mathbf{R}^2$ and radiates a wave u in the presence of a stochastic medium. The wave is governed by an inhomogeneous Helmholtz equation and the Sommerfeld radiation condition,

$$(*)\begin{cases}\left(\Delta + (1+q(x))k_0^2\right)u = f \quad \text{in } \mathbf{R}^2,\\ \lim_{|x|\to\infty}\sqrt{|x|}(\partial_{|x|}u - ik_0u) = 0\\ \qquad\qquad \text{uniformly in } x/|x| \in S^1.\end{cases}$$

The contrast q is a second-order random field with each realization a.s. in $L^2(\Omega)$. We fix $p > 2$ and require $f \in L^p(\Omega)$, ensuring [1] the existence and uniqueness of solution of $(*)$ in $L^p(\Omega)$. Furthermore, the contrast satisfies

1. $\operatorname{supp}(f) \subseteq \operatorname{supp}(q) \subset \Omega$
2. $q(x) > 0 \; a.s, \quad q|_{\partial\Omega} = \eta = \text{const}.$

In particular,

$$q = \eta + \sigma\frac{dW(x)}{dx}\mathbb{1}_\Omega$$

where $dW(x)/dx$ is the formal derivative of the two-sided Brownian sheet. We are here interested in the transfer of information from f to u, in terms of the singular spectrum of the mapping $f \mapsto u$.

2 Forward and normal operator

The system $(*)$ is equivalent with the Lippmann-Schwinger equation

$$\left(I - k_0^2V_q\right)u = V_1 f$$

with the integral operator V_q

$$(V_qf)(x) = \int_\Omega \Phi(x-y)q(y)f(y)dy.$$

For wavenumber k_0 fulfilling

$$k_0 < \left(||q||_{L^\infty(\Omega)}||\Phi(x-\cdot)||_{L^2(\Omega)}\right)^{-1/2}, \qquad (1)$$

the inverse of $I - k_0^2V_q$ is expressible in terms of a convergent Neumann series. Truncating the series at first order defines the mapping $f \mapsto u$ as

$$u = (V_1 + k_0^2V_qV_1)f + O(||q||^2_{L^\infty(\Omega)}).$$

The operator V_1 has a known singular value decomposition [2, 3], and we here consider the normal operator for V_qV_1,

$$\Upsilon = (V_qV_1)^*(V_qV_1).$$

Iterated application of the expansion of the fundamental solution Φ of the Helmholtz operator,

$$\Phi(x-y) = \sum_{n\in\mathbb{Z}} H_n^{(1)}(\kappa)J_n(k_0|y|)e^{in(\arg(x)-\arg(y))},$$

yields [4] a decomposition of Υ to

$$\Upsilon f = \sum_{n\in\mathbb{Z}}\tilde{\lambda}_n\langle f,\psi_n\rangle_{L^2(\Omega)}\psi_n + \sum_{n,m\in\mathbb{Z}}\lambda_{nm}\langle f,\psi_n\rangle_{L^2(\Omega)}\psi_m$$

with 'spectral values' $\tilde{\lambda}_n$, λ_{nm}, and functions ψ_m,

$$\psi_m(\cdot) = J_m(k_0|\cdot|)e^{im\arg(\cdot)},$$
$$\lambda_{nm} = (2\pi R)^2\sigma H_n^{(1)}(\kappa)\xi_m\left|H_m^{(1)}(\kappa)\right|^2H_m^{(2)}(\kappa)J_m(\kappa),$$
$$\tilde{\lambda}_n = (2\pi R)^2\eta\beta_n(\kappa)\left|H_n^{(1)}(\kappa)\right|^4J_n(\kappa).$$

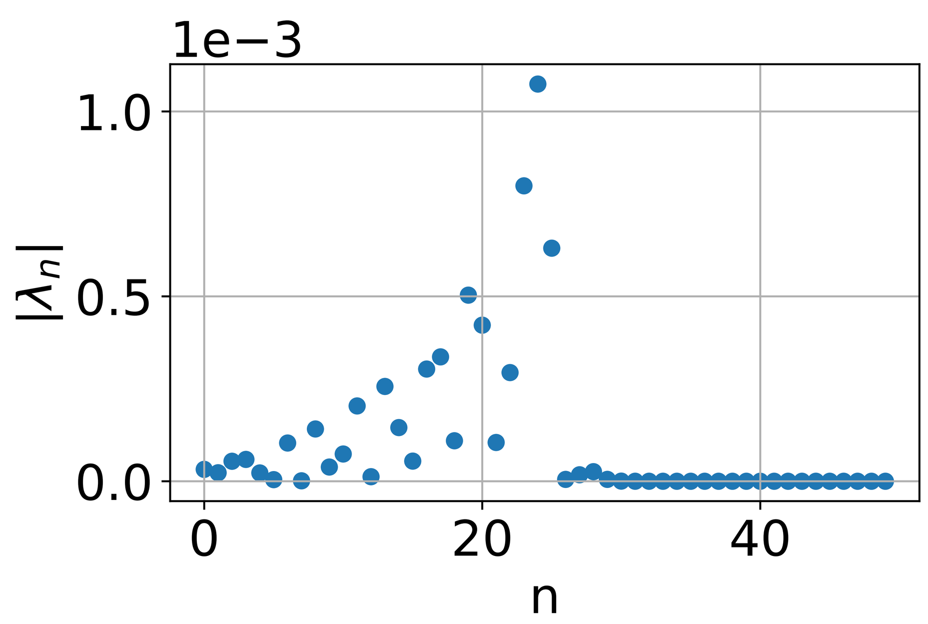
Here

$$\beta_n(\kappa) = \int_0^R rJ_m(k_0r)dr\delta_{n,m}$$

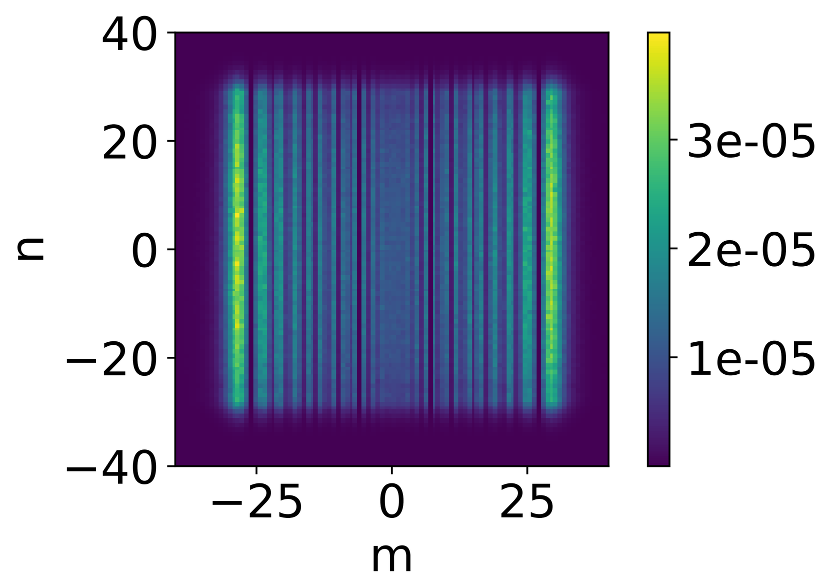
and ξ_m are normally distributed $\mathcal{N}(0, ||\psi_m||^2_{L^2(\Omega)})$. In the second term of the decomposition of Υ, we discover 'spectral leakage.'

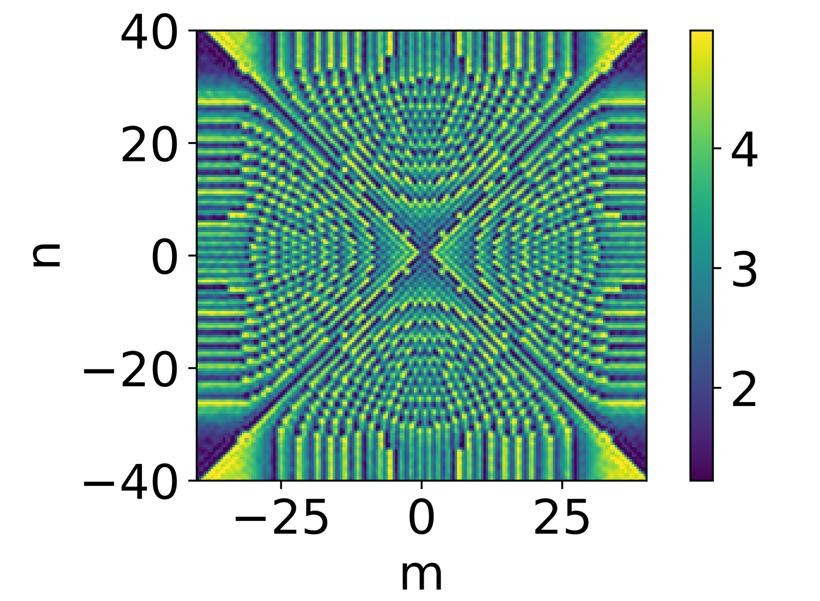
3 Numerical results

With $k_0 = 2\pi$ and $\Omega = \{x \in \mathbf{R}^2,\ |x| < 5\}$, the values $\eta = \sigma = 0.001$ ensure the validity of (1). In this case the values $|\tilde{\lambda}_n|$ are as shown below.



Furthermore, here are the mean magnitude and mean complex argument, respectively, of 400 realizations of λ_{nm}:





This does not yield a spectral decomposition of the operator V_qV_1. However, for any given source f it is possible to numerically estimate the singular values of V_qV_1 utilizing the normal equations.

4 Conclusion

The source-to-field measurement operator for the inhomogeneous Helmholtz equation with random wavenumber has a singular spectrum with a non-trivial random component inducing spectral leakage. This not only influences the magnitude of the spectral values but also causes a non-uniform spatial frequency phase-shift in the measurement. These effects combine to make the inverse source/inverse medium problem even more ill-posed in the presence of a random medium.

Bibliography

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